Title:**Hessian Eigenmaps: New Locally Linear Embedding Techniques for High-Dimensional Data**

Author(s):**David L. Donoho and Carrie Grimes**

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Abstract:

We describe a method to recover the underlying parametrization of scattered data (m_i) lying on a manifold M embedded in high-dimensional Euclidean space. The method, *Hessian-based Locally Linear Embedding* (HLLE), derives from a conceptual framework of *Local Isometry* in which the manifold M, viewed as a Riemannian submanifold of the ambient Euclidean space $\mathbb{R}^n$, is locally isometric to an open, connected subset Θ of Euclidean space $\mathbb{R}^d$. Since Θ does not have to be convex, this framework is able to handle a significantly wider class of situations than the original Isomap algorithm.

The theoretical framework revolves around a quadratic form $\mathcal{H}(f) = \int_M ||H_f(m)||_F^2 dm$ defined on functions $f : M \mapsto \mathbb{R}$. Here Hf denotes the Hessian of f, and $\mathcal{H}(f)$ averages the Frobenius norm of the Hessian over M. To define the Hessian, we use orthogonal coordinates on the tangent planes of M.

The key observation is that, if M truly is locally isometric to an open connected subset of $\mathbb{R}^d$, then $\mathcal{H}(f)$ has a $(d+1)$-dimensional nullspace, consisting of the constant functions and a d-dimensional space of functions spanned by the original isometric coordinates. Hence, the isometric coordinates can be recovered up to a linear isometry.

Our method may be viewed as a modification of the Locally Linear Embedding and our theoretical framework as a modification of the Laplacian Eigenmaps framework, where we substitute a quadratic form based on the Hessian in place of one based on the Laplacian.