Title:
**Continuous Curvelet Transform: II. Discretization and Frames**

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Abstract:

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We develop a unifying perspective on several decompositions exhibiting directional parabolic scaling. In each decomposition, the individual atoms are highly anisotropic at fine scales, with effective support obeying the parabolic scaling principle $length \approx width^2$. Our comparisons allow to extend Theorems known for one decomposition to others.

We start from a Continuous Curvelet Transform $f \mapsto \Gamma_f(a, b, \theta)$ of functions $f(x_1, x_2)$ on $R^2$, with parameter space indexed by scale $a > 0$, location $b \in R^2$, and orientation $\theta$. The transform projects $f$ onto a curvelet $\gamma_{ab\theta}$, yielding coefficient $\Gamma_f(a, b, \theta) = \langle f, \gamma_{ab\theta} \rangle$; the corresponding curvelet $\gamma_{ab\theta}$ is defined by parabolic dilation in polar frequency domain coordinates. We establish a reproducing formula and Parseval relation for the transform, showing that these curvelets provide a continuous tight frame.

The CCT is closely related to a continuous transform introduced by Hart Smith in his study of Fourier Integral Operators. Smith's transform is based on true affine parabolic scaling of a single mother wavelet, while the CCT can only be viewed as true affine parabolic scaling in euclidean coordinates by taking a slightly different mother wavelet at each scale. Smith's transform, unlike the CCT, does not provide a continuous tight frame. We show that, with the right underlying wavelet in Smith's transform, the analyzing elements of the two transforms become increasingly similar at increasingly fine scales.

We derive a discrete tight frame essentially by sampling the CCT at dyadic intervals in scale $a_j = 2^{-j}$, at equispaced intervals in direction, $\theta_{j,\ell} = 2\pi 2^{-j/2}\ell$, and equispaced sampling on a rotated anisotropic grid in space. This frame is a complexification of the 'Curvelets 2002' frame constructed by Emmanuel Candès et al. [**?**, **?**, **?**]. We compare this discrete frame with a composite system which at coarse scales is the same as this frame but at fine scales is based on sampling Smith's transform rather than the CCT. We are able to show a very close approximation of the two systems at fine scales, in a strong operator norm sense.

Smith's continuous transform was intended for use in forming molecular decompositions of Fourier Integral Operators (FIO's). Our results showing close approximation of the curvelet frame by a composite frame using true affine paraboblic scaling at fine scales allow us to cross-apply Smith's results, proving that the discrete curvelet transform gives sparse representations of FIO's of order zero. This yields an alternate proof of a recent result of Candès and Demanet about the sparsity of FIO representations in discrete curvelet frames.