Title:

Sparse Nonnegative Solution of Underdetermined Linear Equations by Linear Programming

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Abstract:

Consider an underdetermined system of linear equations $y = Ax$ with known $d \times n$ matrix A and known y. We seek the sparsest nonnegative solution, i.e. the nonnegative x with fewest nonzeros satisfying $y = Ax$. In general this problem is NP-hard. However, for many matrices A there is a threshold phenomenon: if the sparsest solution is sufficiently sparse, it can be found by linear programming.

In classical convex polytope theory, a polytope P is called *k-neighborly* if every set of k vertices of P span a face of P. Let a_j denote the j-th column of A, $1 \leq j \leq n$, let $a_0 = 0$ and let P denote the convex hull of the a_j. We say P is *outwardly k-neighborly* if every subset of k vertices *not including* 0 spans a face of P. We show that outward k-neighborliness is completely equivalent to the statement that, whenever $y = Ax$ has a nonnegative solution with at most k nonzeros, it is the nonnegative solution to $y = Ax$ having minimal sum.

Using this and classical results on polytope neighborliness we obtain two types of corollaries. First, because many $\lfloor d/2 \rfloor$-neighborly polytopes are known, there are many systems where the sparsest solution is available by convex optimization rather than combinatorial optimization — provided the answer has fewer nonzeros than half the number of equations. We mention examples involving incompletely-observed Fourier transforms and Laplace transforms.

Second, results on classical neighborliness of high-dimensional randomly-projected simplices imply that, if A is a typical uniformly-distributed random orthoprojector with $n = 2d$ and n large, the sparsest nonnegative solution to $y = Ax$ can be found by linear programming provided it has fewer nonzeros than 1/8 the number of equations.

We also consider a notion of weak neighborliness, in which the overwhelming majority of k-sets of a_j's not containing 0 span a face. This implies that most nonnegative vectors x with k nonzeros are uniquely determined by $y = Ax$. As a corollary of recent work counting faces of random simplices, it is known that most polytopes P generated by large n by $2n$ uniformly-distributed orthoprojectors A are weakly k-neighborly with $k \approx .558n$. We infer that for most n by $2n$ underdetermined systems having a sparse solution with fewer nonzeros than roughly half the number of equations, the sparsest solution can be found by linear programming.