Fast Solution of ℓ_1-norm Minimization Problems When the Solution May be Sparse

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October 2006

Abstract

The minimum ℓ_1-norm solution to an underdetermined system of linear equations $y = Ax$, is often, remarkably, also the sparsest solution to that system. This sparsity-seeking property is of interest in signal processing and information transmission. However, general-purpose optimizers are much too slow for ℓ_1 minimization in many large-scale applications.

The Homotopy method was originally proposed by Osborne *et al.* for solving noisy overdetermined ℓ_1-penalized least squares problems. We here apply it to solve the noiseless underdetermined ℓ_1-minimization problem min $\|x\|_1$ subject to $y = Ax$. We show that Homotopy runs much more rapidly than general-purpose LP solvers when sufficient sparsity is present. Indeed, the method often has the following *k-step solution property*: if the underlying solution has only k nonzeros, the Homotopy method reaches that solution in only k iterative steps. When this property holds and k is small compared to the problem size, this means that ℓ_1 minimization problems with k-sparse solutions can be solved in a fraction of the cost of solving one full-sized linear system.

We demonstrate this k-step solution property for two kinds of problem suites. First, incoherent matrices A, where off-diagonal entries of the Gram matrix $A^T A$ are all smaller than M. If y is a linear combination of at most $k \leq (M^{-1} + 1)/2$ columns of A, we show that Homotopy has the k-step solution property. Second, ensembles of $d \times n$ random matrices A. If A has iid Gaussian entries, then, when y is a linear combination of at most $k < d/(2\log(n)) \cdot (1 - \epsilon_n)$ columns, with $\epsilon_n > 0$ small, Homotopy again exhibits the k-step solution property with high probability. Further, we give evidence showing that for ensembles of $d \times n$ partial orthogonal matrices, including partial Fourier matrices, and partial Hadamard matrices, with high probability, the k-step solution property holds up to a dramatically higher threshold k, satisfying $k/d < \hat{\rho}(d/n)$, for a certain empirically-determined function $\hat{\rho}(\delta)$.

Our results imply that Homotopy can efficiently solve some very ambitious large-scale problems arising in stylized applications of error-correcting codes, magnetic resonance imaging, and NMR spectroscopy. Our approach also sheds light on the evident parallelism in results on ℓ_1 minimization and Orthogonal Matching Pursuit (OMP), and aids in explaining the inherent relations between Homotopy, LARS, OMP, and Polytope Faces Pursuit.

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