Title: **Uncertainty Principles and Ideal Atomic Decomposition**

Author(s): **David L. Donoho and Xiaoming Huo**

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Abstract:

Suppose a discrete-time signal $S(t)$, $0 \leq t < N$, is a superposition of atoms taken from a combined time/frequency dictionary made of spike sequences $1_{\{t=\tau\}}$ and sinusoids $\exp\{2\pi iwt/N)/\sqrt{N}$. Can one recover, from knowledge of S alone, the precise collection of atoms going to make up S? Because every discrete-time signal can be represented as a superposition of spikes alone, or as a superposition of sinusoids alone, there is no unique way of writing S as a sum of spikes *and* sinusoids in general.

We prove that if S is representable as a *highly sparse* superposition of atoms from this time/frequency dictionary, then there is only one such highly sparse representation of S, and it can be obtained by solving the *convex* optimization problem of minimizing the ℓ^1 norm of the coefficients among all decompositions. Here "highly sparse" means that $N_t + N_w < \sqrt{N}/2$ where N_t is the number of time atoms, N_w is the number of frequency atoms, and N is the length of the discrete-time signal.

Related phenomena hold for functions of a real variable. We prove that if a function $f(\theta)$ on the circle $[0, 2\pi)$ is representable by a sufficiently sparse superposition of wavelets and sinusoids, then there is only one such sparse representation; it may be obtained by minimum ℓ^1 norm atomic decomposition. The condition "sufficiently sparse" means that the number of wavelets at level j plus the number of sinusoids in the j-th dyadic frequency band are together less than a constant times $2^{j/2}$.

Parallel results hold for functions of two real variables. If a function $f(x_1, x_2)$ on R^2 is a sufficiently sparse superposition of wavelets and ridgelets, there is only one such decomposition and minimum ℓ^1-norm decomposition will find it. Here "sufficiently sparse" means that the total number of wavelets and ridgelets at level j is less than a certain constant times $2^{j/2}$.

Underlying these results is a simple ℓ^1 uncertainty principle which says that if two bases are mutually incoherent, no nonzero signal can have a sparse representation in both bases simultaneously.

The results have idealized applications to bandlimited approximation with gross errors, to error-correcting encryption, and to separation of uncoordinated sources.